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Improving Performance of Direction of Arrival Estimation Using Sparse Arrays in Smart Antenna System

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ABSTRACT: *This paper indicates importance of using sparse arrays (SA) in direction of arrival (DOA) estimation algorithms in smart antenna system (SAS). Analytical study of sparse arrays is introduced, which include coprime array, extended coprime array, nested array, coprime array with compressed interelement spacing (CACIS), and coprime array with displaced subarrays (CADiS).Paper evaluates these sparse arrays using their difference coarray equivalence and derives the analytical expressions of the coarray aperture, the achievable number of unique lags, the maximum number of consecutive lags and degree of freedom (DOF). Compared to uniform arrays with* (M) *sensors, sparse arrays increase the degree of the freedom from*$O(M)$*to* $O(M^2)$*. For comparison of performance of these sparse arrays, numerical example is introduced, where the results indicate that nested array structure provides coarray with unique lags (that are all consecutive), which are larger than that of prototype and extended coprime. Results also indicate that the CACIS structure yields flexibility in trade-off between unique lags and consecutive lags, whereas the CADiS structure allows the minimum interelement spacing to be much larger than the typical half-wavelength requirement, but at the expense of a decrease in consecutive lags. Furthermore, the nested CADiS slightly outperform the nested CACIS due to the higher number of consecutive lags achieved. We propose the scheme for DOA estimation using suitable sparse arrays with SS-MUSIC or LASSO algorithms. According to results, we can choose suitable sparse array and DOA estimation algorithm in SAS depending to the radio situation and the purpose of this SAS. All mentioned arrays and algorithms are simulated using MATLAB. Results of simulations support the theoretical expressions.*

Keywords: *Sparse Arrays (SA), Coprime Arrays, Nested Arrays, CACIS Structure, CADiS Structure.*

1. Introduction

The Smart Antenna System (SAS) embeds the antenna elements and the digital signal processing unit which enables it to form a beam for a desired direction taking into account the multipath signal components. Hence, signal to interference and noise ratio (SINR) can be improved due to the nulls produced towards the interferers in the direction of signal of noninterest (SonI) and the overall spectrum efficiency can be increased [1].

It is common in practice that the number of sources to be estimated is larger than the number of sensors in the array. However, the degree of freedom (DOF) of the conventional source estimation algorithms is limited by the number of sensors. In general, an array antenna with M physical sensors can identify up to $M-1$ sources. To detect more sources, additional sensors are required to increase the achievable DOF, which leads to an increase in complexity. Therefore, an active research topic has been focused on how to increase the DOF for source estimation [2, 3].

Sparse arrays (SAs) open a new approach to sensor array processing due to the higher degree of freedom offered in the difference-coarray domain. Coprime arrays and nested arrays are examples of sparse arrays obtained from a union of two uniform linear arrays (ULAs) with different interelement spacing. The increased degree of freedom has been used to identify $O(M^2)$ sources from only M sensors [4, 5, 6].

In addition to coprime and nested arrays, generalized configurations of the coprime array concept are considered, which comprise two operations. The first operation is the compression of the interelement spacing of one subarray in the coprime array by a positive integer. The resulting coarray structure is referred to as coprime array with compressed interelement spacing (CACIS). The second operation introduces a displacement between the two subarrays, yielding a coprime array with displaced subarrays (CADiS) [2].

Using sparse arrays in SAS to increase the DOF without any increase of physical sensors has been recently the objective of many researches. Coprime array for SS-MUSIC algorithm, and the modification of spatial smoothing step were introduced in [4], but without studying the nested arrays. While nested array with SS-MUSIC algorithm was presented in [7], but without considering coprime arrays. Nested array was also studied in [8], where super nested array was proposed in order to reduce the mutual coupling between sensors, however, coprime and extended coprime arrays were not studied. Extended coprime array with SS-MUSIC algorithm was studied in [9] taking into account the mutual coupling between sensors, but without studying nested arrays and LASSO algorithm. Sparse arrays were studied in [2] and the concept of generalized coprime array was introduced, which involves two operations, producing the CACIS and CADiS configurations, where the performance of the two configurations was evaluated and compared, nevertheless, extended coprime arrays were not considered.

Although these sparse array antennas have been separately reported in the literature, this paper is more comprehensive. Namely, the main contributions in this paper lie in the analytical study of the types and structures of sparse array antennas, *i.e.* coprime, extended coprime, nested array, CACIS, and CADiS structures, as well as the study of DOA estimation using SS-MUSIC and Lasso algorithms and comparing between them using MATLAB environment. In addition, comprehensive performance comparison of these types and structures is achieved(by introducing numerical example) in terms of coarray aperture, unique and consecutive lags, DOF (resolution and number of resolvable signals), and typical half-wavelength requirement for DOA estimation using SS-MUSIC and LASSO algorithms. Furthermore, this paper proposes suitable schemes for DOA estimation using sparse arrays for SS-MUSIC and LASSO algorithms and according to results, suitable sparse array and algorithm in SAS can be chosen depending on the radio situation and the purpose of this SAS.

2. Sparse Arrays

Nested arrays and Coprime arrays are examples of sparse arrays obtained from a union of two uniform linear arrays (ULAs) with different interelement spacing. The increased degree of freedom has been used to identify $O(M^2)$ sources from only M sensors [2, 4, 5].

2.1 Coprime Arrays

As a new concept for array geometry, coprime arrays use two uniform linear arrays (ULAs) with coprime antenna element numbers and coprime interelement distances to achieve a high resolution of DOA estimation and reduce mutual coupling influence [10].

The coprime array consists of two ULAs, where first subarray (ULA1) has N elements with M being the interelement spacing, and the unit interelement spacing $d = \frac{\lambda}{2}$, where λ denotes the wavelength. The second subarray (ULA2) has M elements with N being the interelement spacing (where$N > M$, N and$M$ are integer and coprime numbers). Therefor this array is called the coprime array. Fig (1) indicates the prototype coprime array configuration. Because the two subarrays share the first sensor at the zeroth position, the total number of the sensors used in the coprime array is $(M + N - 1)$. Note that the minimum interelement spacing in this coprime array is $d = \frac{\lambda}{2}$ [2, 10, 11].

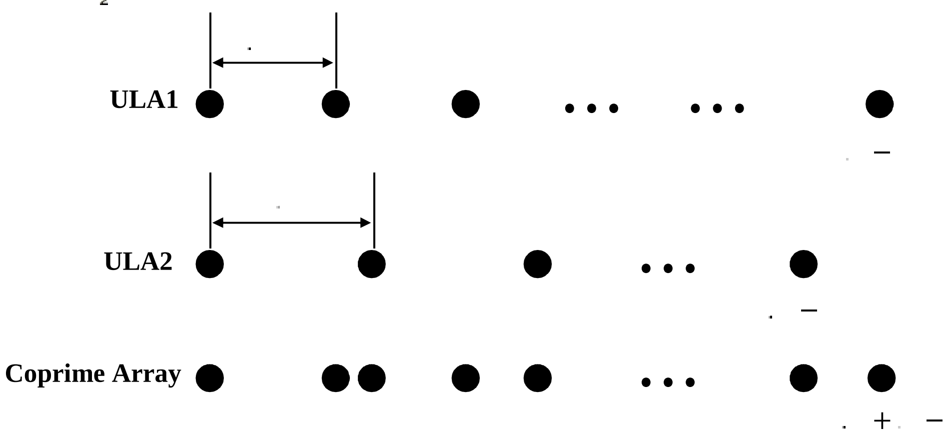

Fig. 1: The prototype coprime array configuration.

The array sensors are positioned at [2]

$$\mathbb{P} = \{M \quad | 0 \leq n \leq N - 1\} \cup \{N \quad | 0 \leq m \leq M - 1\} \quad (1)$$

where $\mathbf{p} = [p_1, \dots, p_{M+N-1}]^T$are the positions of the array sensors where $p_i \in \mathbb{P}, i = 1, \dots, M + N - 1$, and the first sensor is assumed as the reference, *i.e.*$p_1 = 0$.

From a pair of antennas located at the ith and kth positions in $\mathbf{P}$, the correlation $\mathrm{E}[x_i(t)x_k^*(t)]$, where $*$denotes complex conjugation, yields the (i, k)th entry within lag $p_i - p_k$. As such, all the available values of i and k, where$1 \leq i \leq M + N - 1$ and $1 \leq k \leq M + N - 1$, yield virtual sensors of the following difference coarray [2]:

$$\mathbb{C}_P = \{\mathbf{z} | \mathbf{z} = \mathbf{u} - \mathbf{v}, \mathbf{u} \in \mathbb{P}, \mathbf{v} \in \mathbb{P}\} \qquad (2)$$

Using a part or the entire set of the distinct auto-correlation terms in set $\mathbb{C}_P$, instead of the original array, to perform DOA estimation, we can increase the number of detectable sources by the array. The maximum DOF is determined by the number of unique elements in the following set [2]:

$$\mathbb{L}_P = \{l_P | l_P d \in \mathbb{C}_P\} \qquad (3)$$

To gain more insights about the difference coarrays, we separately consider the self-differences of the two subarrays and their cross-differences. The self-difference in the coarray has positions [2]:

$$\mathbb{L}_s = \{l_s | l_s = M\ \} \cup \{l_s | l_s = N\ \} \qquad (4)$$

and the corresponding mirrored positions $\mathbb{L}_s^- = \{-l_s |\ l_s \in \mathbb{L}_s\}$, whereas the cross-difference has positions:

$$\mathbb{L}_c = \{l_c | l_c = N\ \ - M\ \} \qquad (5)$$

and the corresponding mirrored positions $\mathbb{L}_c^- = \{-l_c |\ l_c \in \mathbb{L}_c\}$ for $0 \leq n \leq N - 1$ and $0 \leq m \leq M - 1$. Consequently, the full set of lags in the virtual array is given by,

$$\mathbb{L}_P = \mathbb{L}_s \cup \mathbb{L}_s^- \cup \mathbb{L}_c \cup \mathbb{L}_c^- \qquad (6)$$

Set $\mathbb{L}_P$ contains all unique differences or lags, which comprise set of consecutive coarray lags without holes and is called consecutive differences or lags.

The coprime array produces a coarray that has both redundancy and holes. Redundancy is repeated values of lags, which must be removed for getting all lags to be unique [12, 13].

The number of elements in maximum central ULA segment of its difference coarray is called the uniform degree of freedom or Uniform DOF [8].

Coprime array concept is generalized with two operations. The first operation is the compression of the interelement spacing of one constituting subarray in the coprime array by a positive integer. The resulting coarray structure is referred to as coprime array with compressed interelement spacing (CACIS). The second operation introduces a displacement between the two subarrays, yielding a coprime array with displaced subarrays (CADiS) [2].

2.2 Extended Coprime Array

The coprime array produces a coarray that has both redundancy and holes. For reducing the holes in coarray and getting larger number of consecutive differences or lags, extended coprime array [12] can be used.

As shown in Fig (2), the extended coprime array configuration consists of two ULAs, where ULA1 has N elements with M being the interelement spacing, whereas ULA2 has $2M$elements with N being the interelement spacing [3, 5,12].

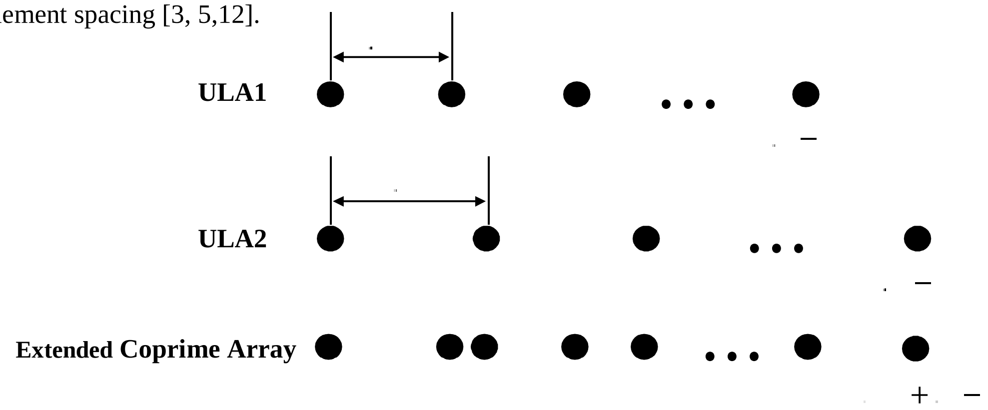

Fig. 2: The Extended Coprime Array configuration.

Because the two subarrays share the first sensor at the zeroth position, the total number of the sensors used in the extended coprime array is $(2M + N - 1)$. Note that the minimum interelement spacing in this coprime array is $d = \frac{\lambda}{2}$.

The sensor locations for extended coprime arrays are [5]:

$$\mathbb{S} = \{0, M, \cdots, (N-1)M, N, \cdots, (2M-1)N\} \qquad (7)$$

The difference coarrays for coprime arrays have a long ULA segment [5]:

$$\mathbb{U} = \{0, \pm 1, \cdots, \pm (M \quad + M - 1)\} \qquad (8)$$

and some missing elements (holes) outside $\mathbb{U}$.

2.3Coprime Array with Compressed Interelement Spacing (CACIS)

Unlike the prototype coprime array, an integer compression factor p is introduced for changing the interelement spacing of one subarray, as well as, the condition that $N > M$ is no longer assumed. Assume that M can be expressed as a product of two positive integers p and $\breve{M}$, *i.e.* [2]

$$M = p\breve{M} \qquad (9)$$

for some p that takes a value between 2 and M. It is easy to confirm that $\breve{M}$ and N are also coprime since M and N do not have common factors other than unity.

As shown in Fig(3), the CACIS configuration consists of two ULAs, where ULA1 has N elements with $\breve{M}d$ being the interelement spacing, whereas ULA2 has M elements with N being the interelement spacing [2].

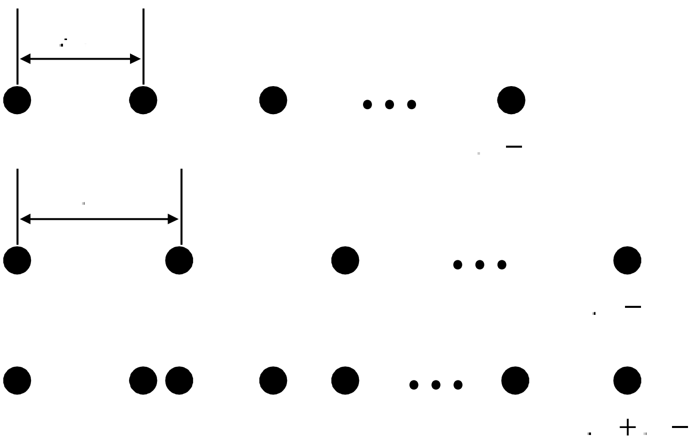

Fig. 3: The CACIS configuration.

Note that the minimum interelement spacing in this array remains $\lambda/2$. Note also that all arrays consist of the same $M + N - 1$ physical antenna sensors and their aperture is $\big((M-1)N\big)d$, regardless of the value of p [2]. It is shown that the variation of the coprime array configuration, which is called Extended Coprime Array, is a special case of the CACIS configuration by choosing $p = 2$ [2, 3, 12].

In this array configuration, the self-lags of the two subarrays are given by the following set[2]:

$$\tilde{\mathbb{L}}_s = \{\tilde{l}_s \mid \tilde{l}_s = \breve{M}n\} \cup \{\tilde{l}_s \mid \tilde{l}_s = N \quad\} \qquad (10)$$

and the corresponding mirrored positions $\tilde{\mathbb{L}}_s^-$, whereas the cross-lags between the two subarrays are given by:

$$\tilde{\mathbb{L}}_c = \{\tilde{l}_c \mid \tilde{l}_c = N \quad - \breve{M}n\} \qquad (11)$$

and the corresponding $\tilde{\mathbb{L}}_c^-$, where $0 \le n \le N-1$ and $0 \le m \le M-1$

When $\breve{M} = 1$, structure of CACIS becomes Nested CACIS structure, which provides the highest number of the unique and consecutive lags (virtual sensors) [2].

2.4Coprime Array with Displaced Subarrays (CADIS)

By introducing a proper displacement between the two collinearly located uniform linear subarrays, the new coprime array structure achieves a larger minimum interelement spacing, a higher number of unique lags, and a larger virtual array aperture. However, the number of consecutive lags is reduced because the positive and negative lags are no longer connected. This new coprime array structure is called Coprime Array with Displaced Subarrays (CADiS), as shown in Fig(4) [2, 14].

Similar to the CACIS configuration, M and N are coprime. The N-element subarray has an interelement spacing of $\tilde{M}d$, and the $(M-1)$-element subarray has an interelement spacing of N .

The difference to the CASIS structure lies in the fact that these two subarrays in the CADiS structure are placed collinearly with the closest spacing between the two subarrays set to L , where $L \geq \min\{\tilde{M}, N\}$ [2].

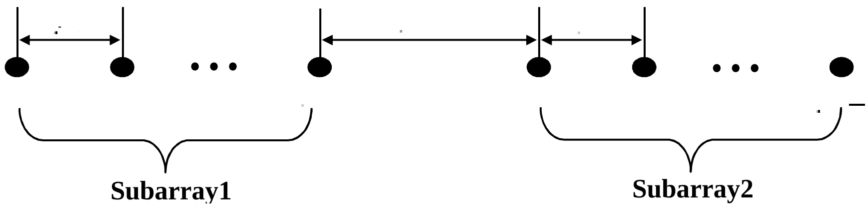

Fig. 4: The CADiS configuration.

Note that the minimum interelement spacing in the CADiS is $\min\{\tilde{M}, N\}d$, as compared to d in the CACIS structure. In addition, the total array aperture of the CADiS is $((N-1)\tilde{M} + (M-2)N + L)d$, which is much larger than the $(M-1)N$ of the CACIS. In practical application, however, a small value of displacement L should be chosen to avoid false peaks [2].

In this array configuration, the self-lags of the two subarrays are given by the following set[2]:

$$\bar{\mathbb{L}}_s = \{\bar{l}_s \mid \bar{l}_s = N \quad\} \cup \{\bar{l}_s \mid \bar{l}_s = \tilde{M}n\} \qquad (12)$$

and the corresponding mirrored positions $\bar{\mathbb{L}}_s^-$, whereas the cross-lags between the two subarrays are given by:

$$\bar{\mathbb{L}}_c = \{\bar{l}_c \mid \bar{l}_c = \tilde{M}(N-1) + N \quad - \tilde{M}n + L\} \qquad (13)$$

and the corresponding$\bar{\mathbb{L}}_c^-$, where$0 \leq m \leq M-2$ and$0 \leq n \leq N-1$.

When $\tilde{M} = 1$, structure of CADiS becomes Nested CADiS structure, which provides the highest number of the unique and consecutive lags (virtual sensors) [2].

2.5Nested Arrays

The nested structure is referred to a structure consisting of two uniform linear subarrays, where one subarray has a unit interelement spacing. A nested array is usually designed such that the virtual sensors in the resulting coarray are all contiguous [2, 15].A nested array generates a coarray with no holes [12].

The nested structure, as shown in Fig (5), consists of an inner$N$-element subarray with a unit spacing d (which is also called dense ULA) and an outer M-element subarray with spacing $(N+1)d$ (which is also called sparse ULA). More precisely it is a linear array with sensors locations given by the union of the sets S_{i} $= \{m \ , \ m = 1, 2, \cdots, N\}$ and S_o $= \{n(N+1)d, \ n = 1, 2, \cdots, M\}$ resulting in $2M(N+1)-1$ consecutive lags. The nested array concept does not require a coprimality between N and M [2, 5, 8, 15, 16, 17].

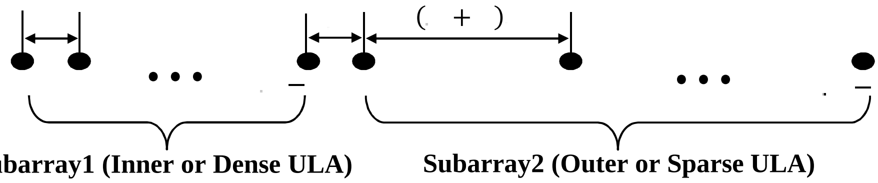

Fig. 5: The nested array configuration.

For nested arrays, the sensor locations are given by [5, 8]:

$$\mathbb{S}_n \quad = \{1, \cdots, N, (N+1), \cdots, M(N+1))\} \qquad (14)$$

The difference coarrays for nested arrays are exactly ULAs, namely,

$$\mathbb{D}_n \quad = \mathbb{U}_n \quad = \{0, \pm 1, \cdots, \pm(M(N+1)-1)\} \qquad (15)$$

Where $\mathbb{D}_n$ and $\mathbb{U}_n$ are of unique and consecutive differences or lags respectively.

It is also important to note that, in the extension of the generalized coprime array framework, different nested array configurations can be defined, by setting $(\breve{M} = 1)$ to the CACIS and CADiS structures. These different nested configurations yield different DOF.

3. Comparison of the Sparse Arrays

For comparison, we enlist in Table (1) analytical expressions of the coarray aperture, the maximum number of unique and consecutive lags for prototype coprime, extended coprime and nested arrays, as well as, CACIS and CADiS structures [2, 3, 5, 12, 15, 16, 17].

Table (1): Analytical expressions of the sparse arrays.

Sparse Array	Coarray Aperture	Maximum Number of Unique Lags	Maximum number of consecutive lags
Prototype Coprime $(\boldsymbol{M+N-1})$ $(\boldsymbol{N} > M)$	$((N-1)M)d$	$M \quad + M + N - 2$	$2(M+N)-1$
Extended Coprime $(\boldsymbol{2\breve{M}+N-1})$ $(\boldsymbol{M = 2\breve{M}})$ $(\boldsymbol{2\breve{M}} > N)$	$((2\breve{M}-1)N)d$	$3\breve{M}N + \breve{M} - N$	$2\breve{M}N + 2\breve{M} - 1$
Nested Array $(\boldsymbol{M+N})$	$(M(N+1)-1)d$	$2M(N+1)-1$	$2M(N+1)-1$
CACIS $(\boldsymbol{M+N-1})$	$((M-1)N)d$	$2M \quad - \breve{M}(N-1) - N$	$2M \quad - 2\breve{M}(N-1) - 1$
CADiS $(\boldsymbol{M+N-1})$ $(\boldsymbol{\breve{M}} > 1)$	$((N-1)\breve{M} + (M-2)N + L)d$ $(L = \breve{M} + N)$	$2M \quad + 2\breve{M} - 1$ $(L = \breve{M} + N)$	$2(M \quad - (\breve{M}-1)(N-2)+1)$ $(L = \breve{M} + N)$
CADiS $(\boldsymbol{M+N-1})$ $(\boldsymbol{\breve{M} = 1})$	$(M \quad)d$ $(L = N+1)$	$2M \quad + 1$ $(L = N+1)$	$2M \quad + 1$ $(L = N+1)$

Table (2) indicates numerical example for comparison of the sparse arrays in terms of values of the coarray aperture, unique and consecutive lags and degree of freedom for DOA estimation using SS-MUSIC and LASSO algorithms, as well as, the typical half-wavelength requirements. This comparison is achieved for sparse array consisting of $M = 6$, $N = 7$, $(M + N - 1) = 12$, with different values of compression factor p and $L = \breve{M} + N$, and unit interelement spacing $d = \lambda/2 = 1$.

It is clear that the structure of extended coprime achieves larger values unique and consecutive lags and DOF than that of prototype coprime. The structure of nested array provides coarray with 83 unique lags, which are all consecutive and larger than that of prototype and extended coprime.

For the structure of CACIS, it is clear that the number of unique and consecutive lags increases as $\breve{M}$ decreases (or as pincreases). When $\breve{M} = 1$, structure of CACIS becomes Nested CACIS structure, which provides the highest number of unique and consecutive lags (virtual sensors) with 71 unique lags, which are all consecutive in the range of [-35, 35], and DOF of 35 which equals the coarray aperture.

Sharing the same property as the prototype and extended coprime array, the CACIS structure provides sparse configurations in which the minimum interelement spacing remains the unit spacing $d = \lambda/2$, which is typically half wavelength. So the CACIS structuredoesn't provide a minimum interelement spacing larger than typical half-wavelength requirements, which is more effectivein many applications where a small interelements spacing is infeasible.

For the structure of CADiS, it is clear that the number of unique lags increases as $\breve{M}$ increases (or as p decreases), whereas the number of the consecutive lags decreases. When $\breve{M} = 1$, structure of CADiS becomes Nested CADiS structure, which provides the highest number of the unique and consecutive lags (virtual sensors) with 85 unique lags, which are all consecutive in the range of [-42, 42], and DOF of 42 which equals the coarray aperture. In case of $\breve{M} > 1$, the number of the consecutive lags, in CADiS structure, is smaller than that of CACIS structure because the positive and negative lags are no longer connected. In case of $\breve{M} > 1$ (non-Nested CADiS), the CADiS structure allows the minimum interelement spacing to be m $\{\breve{M}, N\}d$, which is much larger than the typical half-wavelength requirement, making it useful in applications where a small interelements spacing is infeasible. It is also shown, non-nested CADiS structure, with MUSIC algorithm, provides the lowest number of the consecutive lags (DOF=17, 19), and this structure suffers from significant performance degradation due to the disconnected coarray lags.

Table (2): Comparison of the sparse arrays using numerical example for $(M + N - 1) = 1$.

Sparse Array		Coarray Aperture	Unique Lags	Consecutive Lags	DOF (MUSIC) (LASSO)	Typical Half-Wavelength Requirements
Prototype Coprime $M + N - 1 = 1$		36	53	25	12 26	unit interelement spacing: $d = \lambda/2$
Extended Coprime $M + N - 1 = 1$		35	59	47	23 29	unit interelement spacing: $d = \lambda/2$
Nested Array $(M + N = 1\ )$		41	83	83	41 41	unit interelement spacing: $d = \lambda/2$
CACIS $M + N - 1 = 1$	$p = 2$, $\breve{M} = 3$	35	59	47 [-23, 23]	23 29	unit interelement spacing: $d = \lambda/2$
CACIS $M + N - 1 = 1$	$p = 3$, $\breve{M} = 2$	35	65	59 [-29, 29]	29 32	unit interelement spacing: $d = \lambda/2$
CACIS $M + N - 1 = 1$	$p = 6$, $\breve{M} = 1$	35	71	71 [-35, 35]	35 35	unit interelement spacing: $d = \lambda/2$
CADiS $M + N - 1 = 1$	$p = 2$, $\breve{M} = 3$	56	89	(33, 33) [-44,-12]=33 [12,44]=33	17 44	unit interelement spacing: $\min\{\breve{M}, N\}d$ which is larger than unit spacing $d = \lambda/2$ (*i.e.* satisfies requirements)
CADiS $M + N - 1 = 1$	$p = 3$, $\breve{M} = 2$	49	87	(38, 38) [-43,-6]=38[6,43]=38	19 43	unit interelement spacing: $\min\{\breve{M}, N\}d$ which is larger than unit spacing $d = \lambda/2$ (*i.e.* satisfies requirements)
CADiS $M + N - 1 = 1$	$p = 6$, $\breve{M} = 1$	42	85	85 [-42,42]	42 42	unit interelement spacing: $d = \lambda/2$

The CACIS structure yields flexibility in trade-off between unique lags and consecutive lags for effective DOA estimation based on different algorithms, whereas the CADiS structure further allows a larger minimum interelement spacing beyond the typical half-wavelength requirement.

For SS-MUSIC algorithm, the DOF is roughly equal to half of the available consecutive lags of the resulting coarray, whereas, for LASSO algorithm, the DOF is roughly equal to half of the available unique lags of the resulting coarray.

4. ALGORITHMS OF DOA ESTIMATION USING SPARSE ARRAYS

We introduce scheme for estimating DOA using sparse arrays, where unique and consecutive lags are extracted from the resulting difference coarray. Then SS-MUSIC algorithm is applied on available consecutive lags or LASSO algorithm is applied on available unique lags, as shown in Fig(6).

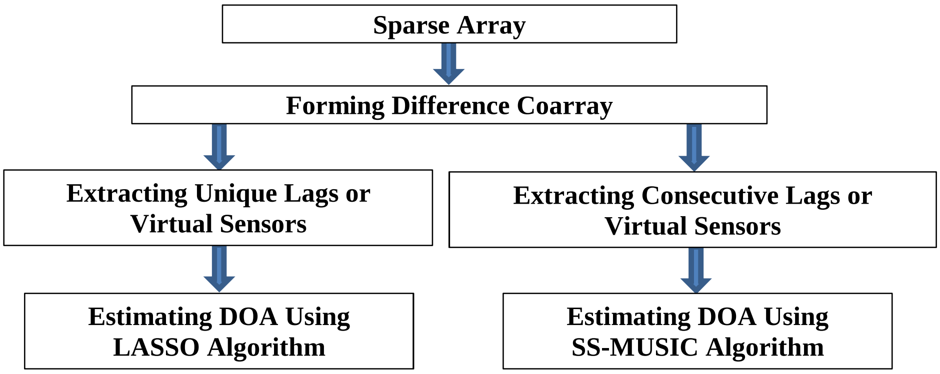

Fig. 6: Scheme for DOA estimation using sparse arrays.

Assume that D uncorrelated signals impinging on the array from angles $\boldsymbol{\Theta} = [\theta_1, \dots, \theta_D]^T$, and their discretized baseband waveforms are expressed as $\mathrm{s_q(k)}$, $k = 1, \dots, K$, for $q = 1, \dots, D$ and K is the number of snapshots. Then, the data vector received at the coprime array is expressed as following [2, 3, 9, 14]:

$$\mathbf{x}(k) = \sum_{q-1}^{D} \mathbf{a}(\theta_q) s_q(k) + \mathbf{n}(k) = \mathbf{A}\ \ (k) + \mathbf{n}(k) \qquad (16)$$

where $\mathrm{a}(\theta_q) = \left[1, e^{j\frac{2\pi p_2}{\lambda} s\ (\theta_q)}, \dots, e^{j\frac{2\pi p_{M+N-1}}{\lambda} s\ (\theta_q)}\right]^T$ is the steering vector of the array corresponding to θ_q, $\mathbf{A} = [\mathbf{a}(\theta_1), \dots, \mathbf{a}(\theta_D)]$, and $\mathbf{s}(k) = [s_1(k), \dots, s_D(k)]^T$. The elements of the noise vector $\mathrm{n(k)}$ are assumed to be independent and identically distributed (i.i.d.) random variables following the complex Gaussian distribution $C\ \ (0, \sigma_n^2 \mathbf{I}_{M+N-1})$.

The covariance matrix of data vector $\mathbf{x}(t)$ is obtained as following [2, 3, 9]:

$$\mathbf{R_x}\ \ = \mathrm{E}[\mathbf{x}(k)\mathbf{x}^H(k)] = \mathbf{A}\mathbf{R_s}\ \mathbf{A}^H + \sigma_n^2 \mathbf{I}_{M+N-1} \qquad (17)$$

where $\mathbf{R_s}\ \ = \mathrm{E}[\mathbf{s}(k)\mathbf{s}^H(k)] = \mathrm{diag}([\sigma_1^2, \dots, \sigma_D^2])$is the source covariance matrix, with σ_q^2 denoting the input signal power of the qth source.

4.1 MUSIC Algorithm of DOA Estimation Using Sparse Arrays

MUSIC represents MUltiple SIgnal Classification and was proposed by Schmidt [18]. Vectorizing matrix $\mathbf{R_x}$ yields the vector $\mathbf{z}$ [2, 3, 9, 14, 16, 19]:

$$\mathbf{z} = \mathrm{vec}(\mathbf{R_x}\ ) = \tilde{\mathbf{A}}\ \mathbf{b} + \sigma_n^2 \hat{\mathbf{I}} = \mathbf{B} \qquad (18)$$

where $\tilde{\mathbf{A}} = [\tilde{\mathbf{a}}(\theta_1), \dots, \tilde{\mathbf{a}}(\theta_D)]$, $\tilde{\mathbf{a}}(\theta_q) = \mathbf{a}^{\square}(\theta_q) \otimes \mathbf{a}(\theta_q)$, $\otimes$ denotes the Kronecker product, $\mathbf{b} = [\sigma_1^2, \dots, \sigma_D^2]^T$, $\hat{\mathbf{I}} = \mathrm{vec}(\mathbf{I}_{M+N-1})$. In addition, $\mathbf{B} = [\tilde{\mathbf{A}}, \hat{\mathbf{I}}]$ and $\mathbf{r} = [\mathbf{b}^T, \sigma_n^2]^T = [\sigma_1^2, \dots, \sigma_D^2, \sigma_n^2]$ are used for notational simplicity.

The vector $\mathbf{z}$ amounts to the received data from a virtual array with an extended coarray aperture whose corresponding steering matrix is defined by $\tilde{\mathbf{A}}$. However, the virtual source signal becomes a single snapshot of $\mathbf{b}$. In addition, the rank of the noise-free covariance matrix of $\mathbf{z}$, $\mathbf{R_z}\ \ = \mathbf{z}\mathbf{z}^H$, is one. As such, the problem is similar to handling fully coherent sources, and subspace-based DOA estimation techniques, such as MUSIC, fail to yield reliable DOA estimation when multiple signals impinge to the array [2, 3, 9, 14].

To overcome this problem, spatial smoothing technique is applied to the covariance matrix so that its rank can be restored. Since spatial smoothing requires a consecutive difference lag set so that every subarray has similar manifold, we extract all the consecutive lag samples of $\mathbf{z}$ and form a new vector $\mathbf{z}_1$. Denote $[-l_\xi, l_\xi]$ as the consecutive lag range in $\mathbb{L}_F$. Then, $\mathbf{z}_1$ can be expressed as following [2, 3, 9, 14, 16, 19]:

$$\mathbf{z}_1 = \tilde{\mathbf{A}}_1 \mathbf{b} + \sigma_n^2 \hat{\mathbf{I}}_1 \qquad (19)$$

where $\tilde{\mathbf{A}}_1$ is identical to the manifold of a uniform linear array (ULA) with $2l_\xi + 1$ sensors located from $-l_\xi d$ to $l_\xi d$. $\tilde{\mathrm{A}}_1$ can be expressed as [19, 20]:

$$\widetilde{\mathbf{A}}_1 = \begin{bmatrix} e^{-jl_\xi d\frac{2\pi}{\lambda}s}\ (\theta_1) & e^{-jl_\xi d\frac{2\pi}{\lambda}s}\ (\theta_2) & \cdots & e^{-jl_\xi d\frac{2\pi}{\lambda}s}\ (\theta_D) \\ e^{-j(l_\xi-1)d\frac{2\pi}{\lambda}s}\ (\theta_1) & e^{-j(l_\xi-1)d\frac{2\pi}{\lambda}s}\ (\theta_2) & \cdots & e^{-j(l_\xi-1)d\frac{2\pi}{\lambda}s}\ (\theta_D) \\ \vdots & \vdots & & \vdots \\ e^{j(l_\xi-1)d\frac{2\pi}{\lambda}s}\ (\theta_1) & e^{j(l_\xi-1)d\frac{2\pi}{\lambda}s}\ (\theta_2) & \cdots & e^{j(l_\xi-1)d\frac{2\pi}{\lambda}s}\ (\theta_D) \\ e^{jl_\xi d\frac{2\pi}{\lambda}s}\ (\theta_1) & e^{jl_\xi d\frac{2\pi}{\lambda}s}\ (\theta_2) & \cdots & e^{jl_\xi d\frac{2\pi}{\lambda}s}\ (\theta_D) \end{bmatrix} \tag{20}$$

and $\tilde{\mathrm{I}}_1$ is $(2l_\xi + 1)\ x\ 1$ a vector of all zeros except a 1 at the $(l_\xi + 1)$th position.

We divide this virtual array into $l_\xi + 1$ overlapping subarrays, each with $l_\xi + 1$ elements, where the i th subarray has sensors located at $(-i + 1 + k)d$, with $k = 0, 1, \dots, l_\xi$ denoting the index of the overlap subarray used in the spatial smoothing. The equivalent received signal vector of the i-th virtual uniform linear subarray can be denoted as $\mathbf{z}_{1i}, i = 1, \dots, l_\xi + 1$ [2, 3].

Calculating the correlation statistics of each subarray received signal $\mathbf{z}_{1i}$ yields the following rank-one covariance matrix [2, 3, 7, 9, 16]:

$$\mathbf{R}_i = \mathbf{z}_{1i}\mathbf{z}_{1i}^H \tag{21}$$

Taking the average of $\mathbf{R}_i$ over all i, we obtain the spatially smoothed covariance matrix [2, 3, 6, 7, 9, 16]:

$$\mathbf{R}'_{\mathbf{Z}} = \frac{1}{l_\xi + 1}\sum_{i=1}^{l_\xi+1}\mathbf{R}_i \tag{22}$$

which yields a full-rank $(l_\xi + 1)$ covariance matrix so that the MUSIC algorithm can be performed for DOA estimation directly. As a result, l_ξ DOF is achieved, which is roughly equals to half of the available consecutive lags of the resulting coarray [2, 9].

4.2 LASSO Algorithm of DOA Estimation Using Sparse Arrays

LASSO represents Least Absolute Shrinkage and Selection Operator and proposed by Robert Tibshirani [21]. To explore the whole aperture of the coarray for DOA estimation, compressive sensing (CS)-based methods such as LASSO can be used, while such methods suffer from basis mismatch effects [2, 14].

Alternatively, equation (18) can be solved using the CS approach. The desired result of $\mathbf{b}$, whose elements are the first D entries of vector $\mathbf{r}$, can be obtained from the solution to the following constrained l_0-norm minimization problem [2, 14, 21]:

$$\hat{\mathbf{r}}^\circ = \arg\min_{\mathbf{r}^\circ} \|\mathbf{r}^\circ\|_\circ \ \text{s.t.} \ \|\bar{\mathbf{z}} - \mathbf{B}^\circ\mathbf{r}^\circ\|_2 < \epsilon \tag{23}$$

where ϵ is a user-specific bound, $\mathbf{B}^\circ = [\mathbf{A}^g, \bar{\mathbf{I}}]$ is a sensing matrix consisting of the searching steering vectors $\mathbf{A}^g$ and $\mathbf{I}$, whereas $\mathbf{r}^\circ$ is the sparse entries in these search grids to be determined.

The objective function of the Lasso algorithm is defined as following [2, 14, 21, 22]:

$$\hat{\mathbf{r}}^\circ = \arg\min_{\mathbf{r}^\circ}\left[\frac{1}{2}\|\bar{\mathbf{z}} - \mathbf{B}^\circ\mathbf{r}^\circ\|_2 + \lambda_t\|\mathbf{r}^\circ\|_1\right] \tag{24}$$

where l_2 the norm in the objective function denotes the ordinary least-squares (OLS) cost function, and l_1 the norm involves the sparsity constraint. In addition, λ_t is a penalty or a nonnegative regularization parameter which can be tuned to trade off the OLS error for the number of nonzero entries (degree of sparsity) in the estimates. This parameter λ_t controls the relative importance between the sparsity of the solution (l_1-norm term) and the fitness to the measurements (l_2-norm term).

4.3 Simulation of SS-MUSIC and LASSO Algorithms Using Sparse Arrays

Depending on values indicated in Table (2), simulation is performed for sparse array with $M = 6$, $N = 7$, $(M + N - 1) = 12$.

Suppose $D = 25$ uncorrelated narrowband signals are considered, which are uniformly distributed in the angular range of $[-60^\circ : 5 : 60^\circ]$, the number of snapshots is $K = 32$ and the signal to noise ratio is $\mathrm{SNR} = 5\mathrm{dB}$.

DOA of signals is estimated when the DOF is bigger than the number of these signals. The DOF depends on the sparse array structure and the DOA estimation algorithm used because SS-MUSIC algorithm uses consecutive lags or virtual sensors, whereas LASSO algorithm uses unique lags or virtual sensors.

4.3.1 Using CACIS Structure

The signals $D = 25$ are received and the DOA is estimated by CACIS structure using SS-MUSIC algorithm for compression factor $p = 3, 6$ (*i.e.*$\breve{M} = 2, 1$) which corresponds to $DOF = 29, 35$ respectively. Whereas using LASSO algorithm, the DOA is estimated for $p = 2, 3, 6$ (*i.e.*$\breve{M} = 3, 2, 1$) which corresponds to $DOF = 29, 32, 35$ respectively, as shown in Table (2).

It is shown that when applying LASSO algorithm with CACIS structure, bigger DOF is achieved as all available unique lags are exploited. While when applying SS-MUSIC algorithm with CACIS structure, the DOF depends only on the available consecutive lags. Therefore, LASSO algorithm outperforms SS-MUSIC algorithm in terms of the degree of freedom (DOF).In addition, the performance improves as the compression factor increases due to the higher unique and consecutive lags achieved.

For performance comparison of SS-MUSIC and LASSO algorithms in terms of DOA resolution, we focus our study on the same case in CACIS structure, $p = 6$ (*i.e.*$\breve{M} = 1$or Nested CACIS), which corresponds to $DOF = 35$. Fig (7) indicates estimation of DOA using SS-MUSIC with Nested CACIS, while Fig (8) indicates estimation of DOA using LASSO with Nested CACIS.

As shown in Fig (7) and Fig (8), it is clear that both SS-MUSIC and LASSO algorithms can estimate the DOA of all signals using Nested CACIS which corresponds to $DOF = 35$. It is also shown that the SS-MUSIC algorithm slightly outperforms LASSO algorithm in terms of DOA resolution.

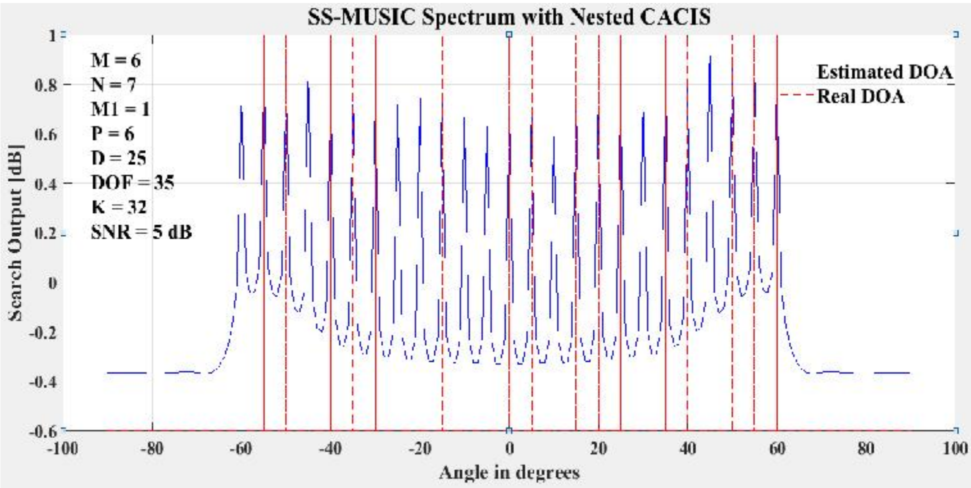

Fig. 7: Estimation of DOA by Nested CACIS (DOF = 35) and Using SS-MUSIC.

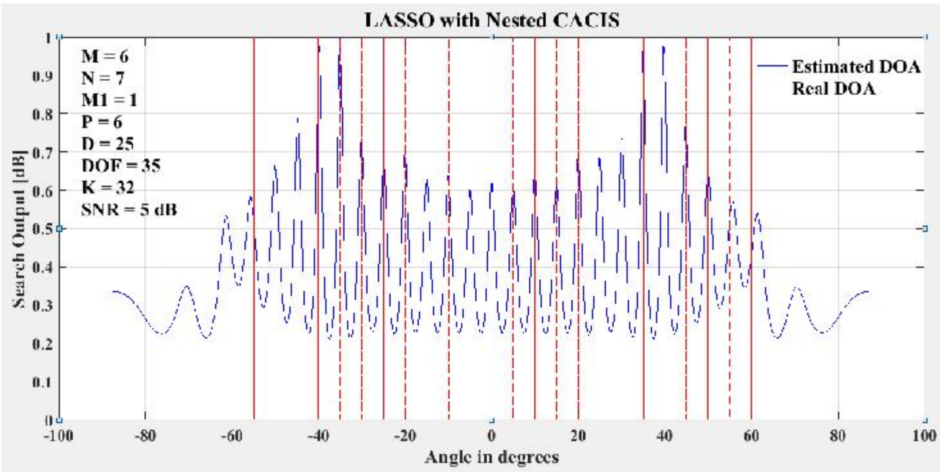

Fig. 8: Estimation of DOA by Nested CACIS (DOF = 35)and Using LASSO.

4.3.2 Using CADiS Structure

The signals $D = 25$ are received and the DOA is estimated by CADiS structure using SS-MUSIC algorithm for compression factor $p = 6$ (*i.e.* $\breve{M} = 1$ or Nested CADiS), which corresponds to $\text{DOF} = 42$. Whereas using LASSO algorithm, the DOA is estimated for $p = 2, 3, 6$, (*i.e.* $\breve{M} = 3, 2, 1$), which corresponds to $\text{DOF} = 44, 43, 42$ respectively, as shown in Table (2).

It is shown that when applying LASSO algorithm with CADiS structure, bigger DOF is achieved as all available unique lags are exploited. While when applying SS-MUSIC algorithm with CADiS structure, the DOF depends only on the available consecutive lags. Therefore, LASSO algorithm outperforms SS-MUSIC algorithm in terms of the degree of freedom (DOF).

For comparison of performance of SS-MUSIC and LASSO algorithms in terms of DOA resolution, we focus our study on the same case in CADiS structure, $p = 6$ (*i.e.* $\breve{M} = 1$ or Nested CADiS), which corresponds to $\text{DOF} = 42$. Fig(9) indicates DOA estimation using SS-MUSIC with Nested CADiS, while Fig(10) indicates DOA estimation using LASSO with Nested CADiS.

As shown in Fig (9) and Fig (10), it is clear that both SS-MUSIC and LASSO algorithms can estimate the DOA of all signals using Nested CADiS which corresponds to $\text{DOF} = 42$. It is also shown that the SS-MUSIC algorithm slightly outperforms LASSO algorithm in terms of DOA resolution.

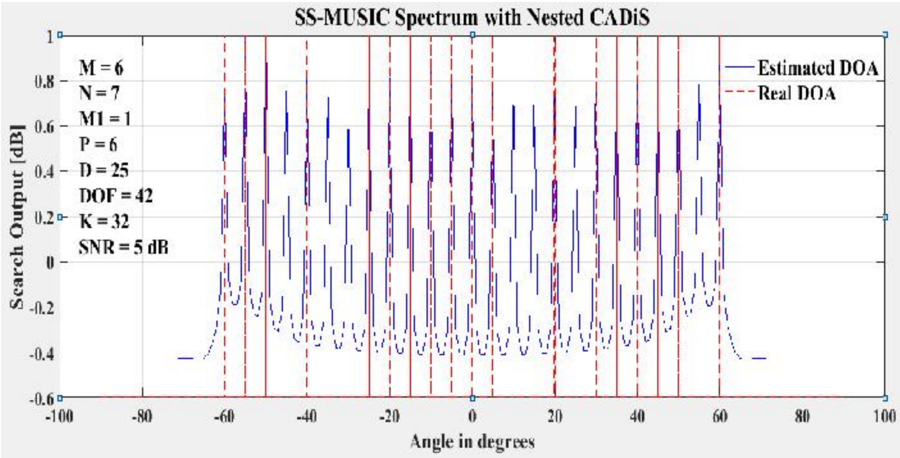

Fig. 9: Estimation of DOA by Nested CADiS (DOF = 42)and Using SS-MUSIC.

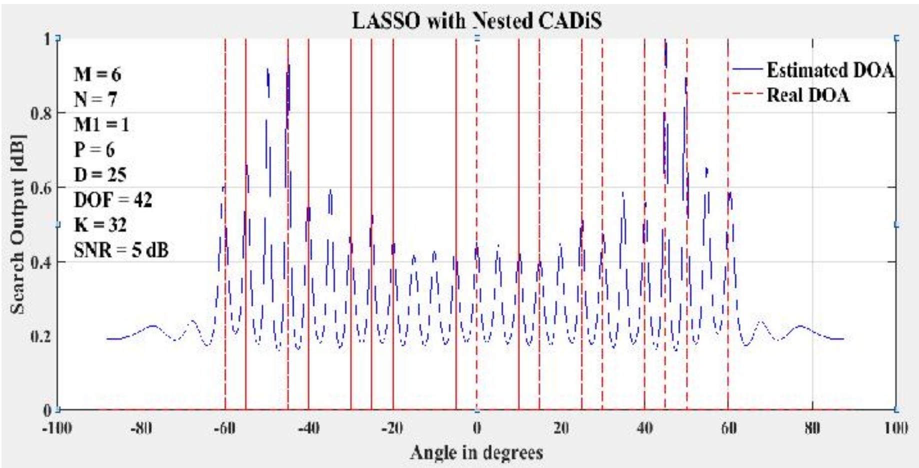

Fig. 10: Estimation of DOA by Nested CADiS (DOF = 42) and Using LASSO.

Comparing the performance of CACIS and CADiS structures (as shown in Fig (7) and Fig (9) for SS-MUSIC algorithm and Fig (8) and Fig (10) for LASSO algorithm), the Nested CADiS structure is better than the Nested CACIS structure due to the higher available DOF, since the Nested CADiS structure yields $\text{DOF} = 42$, whereas the Nested CACIS structure yields $\text{DOF} = 35$.

According to the results, we can choose the suitable sparse array and DOA estimation algorithm in SAS depending on the radio situation and the purpose of this SAS. Table (3) indicates features of resulted coarray for CACIS and CADiS structures and proposed algorithms according to the radio situation and the required DOA high resolution.

If the radio situation is complex (*i.e.* contains many signals) and a high resolution of DOA estimation is required, we should choose either Nested CACIS with SS-MUSIC algorithm or Nested CADiS with SS-MUSIC algorithm because Nested CACIS and Nested CADiS provide maximum number of consecutive lags and SS-MUSIC algorithm slightly outperforms LASSO algorithm in terms of DOA resolution. Nested CACIS structure with SS-MUSIC is preferred here due to the better computational complexity (*i.e.* the smaller).Computational complexity of DOA estimation algorithm increases as the number of virtual elements (sensors) of the sparse array (unique or consecutive lags) increase.

If the radio situation is complex (*i.e.* contains many signals) and a high resolution of DOA estimation is not required, we should choose either Nested CACIS with LASSO or CADiS structure (nested and non-nested CADiS) with LASSO. Nested CACIS structure with LASSO is preferred due to the better computational complexity (*i.e.* the smaller).

If the radio situation is not complex (*i.e.* contains few signals) and a high resolution of DOA estimation is required, we should choose either non-nested CACIS with SS-MUSIC or non-nested CADiS with SS-MUSIC. Non-nested CADiS with SS-MUSIC is preferred due to the better computational complexity (*i.e.* the smaller).

If the radio situation is not complex (*i.e.* contains few signals) and a high resolution of DOA estimation is not required, we should choose either non-nested CACIS with any algorithms (SS-MUSIC or LASSO) or non-nested CADiS with SS-MUSIC. Non-nested CADiS with SS-MUSIC is preferred due to the better computational complexity (*i.e.* the smaller).

In applications where a small interelement spacing is infeasible, non-nested CADiS structure is more effectiveas it allows the minimum interelement spacing to be $\min\{\breve{M}, N\}d$ which is much larger than the typical half-wavelength requirement, but at the expense of a decrease in consecutive lags.

The terms or descriptions "large, larger and largest" are relative and used for comparison between CACIS and CADiS structures.

Table (3): Features of coarray for CACIS and CADiS structures and proposed algorithms according to the radio situation and the required DOA high resolution.

<table>
<tr><td colspan="2" rowspan="2">Features of coarray structure</td><td colspan="2">CACIS structure</td><td colspan="2">CADiS structure</td></tr>
<tr><td>Non-nested CACIS $\breve{M} > 1$</td><td>Nested CACIS $\breve{M} = 1$</td><td>Non-nested CADiS $\breve{M} > 1$</td><td>Nested CADiS $\breve{M} = 1$</td></tr>
<tr><td colspan="2">Coarray aperture</td><td colspan="2">large</td><td>largest</td><td>larger</td></tr>
<tr><td colspan="2">Number of unique lags</td><td>large</td><td>larger</td><td colspan="2">largest</td></tr>
<tr><td colspan="2">Number of consecutive lags</td><td>large</td><td>larger</td><td>non-large</td><td>largest</td></tr>
<tr><td colspan="2">Computational complexity</td><td>large</td><td>larger</td><td>largest (with unique lags)
non-large (with consecutive lags)</td><td>Largest</td></tr>
<tr><td colspan="2">Typical half-wavelength requirements</td><td>unit spacing $d = \lambda/2$</td><td>unit spacing $d = \lambda/2$</td><td>$(\min\{\breve{M}, N\}d)$is larger than unit spacing$d = \lambda/2$ (*i.e.* satisfies requirements)</td><td>unit spacing $d = \lambda/2$</td></tr>
<tr><td>Radio situation</td><td>DOA high resolution</td><td colspan="4">Proposed algorithm</td></tr>
<tr><td rowspan="2">Complex</td><td>Required</td><td>-</td><td>SS-MUSIC</td><td>-</td><td>SS-MUSIC</td></tr>
<tr><td>Non-required</td><td>-</td><td>LASSO</td><td>LASSO</td><td>LASSO</td></tr>
<tr><td>Non-complex</td><td>Required</td><td>SS-MUSIC</td><td>-</td><td>SS-MUSIC</td><td>-</td></tr>
</table>

	Non-required	SS-MUSIC, LASSO	-	SS-MUSIC	-

5. CONCLUSIONS

Importance of using sparse arrays in direction of arrival DOA estimation algorithms in smart antenna system (SAS) has been shown. Analytical study of sparse arrays, which include coprime array, extended coprime array, nested array as well as CACIS and CADiS structures, has been presented. These sparse arrays have been evaluated using their difference coarray equivalence, where analytical expressions have been derived for the coarray aperture, the achievable number of unique lags, the maximum number of consecutive lags, and the degree of freedom (DOF). Compared to uniform arrays with (M) sensors, sparse arrays increase the degree of the freedom from $O(M)$ to $O(M^2)$.

For comparison of performance of these sparse arrays, numerical example has been introduced, where results have indicated that nested array structure provides coarray with unique lags, which are all consecutive and larger than that of prototype and extended coprime. Results have also indicated that the CACIS structure yields flexibility in trade-off between unique lags and consecutive lags, whereas the CADiS structure allows the minimum interelement spacing to be much larger than the typical half-wavelength requirement (in case of$\tilde{M} > 1$ or non-nested CADiS structure), but at the expense of a decrease in consecutive lags. Non-nested CADiS structure with MUSIC algorithmprovides the lowest number of the consecutive lags and this structure suffers from significant performance degradation due to the disconnected coarray lags. Furthermore, the Nested CADiS slightly outperform the Nested CACIS due to the higher number of consecutive lags achieved.

The scheme for DOA estimation using sparse arrays with SS-MUSIC and LASSO algorithms has been proposed. When LASSO is applied, higher DOF is achieved because all available unique lags are exploited, while when SS-MUSIC is applied, lower DOF is achieved as only available consecutive lags are exploited. The SS-MUSIC algorithm slightly outperforms LASSO algorithm in terms of DOA resolution, while LASSO algorithm outperforms SS-MUSIC algorithm in terms of the degree of freedom (DOF). In other words, when applying LASSO algorithm, bigger DOF is achieved as all available unique lags are exploited. According to results, suitable sparse array and DOA estimation algorithm can be chosen in SAS depending on the radio situation and the purpose of this SAS.

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